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EXACT FINITE DIFFERENCE SCHEMES FOR TWO-DIMENSIONAL ADVECTION EQUATIONS

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Many interesting systems in acoustics and fluid dynamics may be mathematically modelled by partial differential equations (PDE) where linear advection and/or non-linear reaction are the dominant effects [1–3]. For two space dimensions, the PDE's take the form

$$u_t + au_x + bu_y = 0, \qquad u_t + au_x + bu_y = u(1-u), \tag{1, 2}$$

where a and b are constants, and the reaction term in equation (2) is taken to be a normal quadratic polynomial in $u(x, y, t)$. Taken as initial value problems, i.e., $u(x, y, 0) = f(x, y)$ given, one will show below that both PDE's can be solved exactly for

$$-\infty < x < +\infty, \qquad -\infty < y < +\infty, \qquad t > 0. \tag{3}$$

However, a major reason for considering these equations is that they provide the opportunity for studying finite difference schemes used to numerically integrate more complex sets of PDE's [4, 5].

The main purpose of this letter is to derive exact finite difference schemes for equations (1) and (2). These exact schemes can then be used as bench marks for purpose of comparison with finite difference schemes constructed using the standard methods [2, 4, 5]. This work extends the results previously reported in references [6, 7]. As a reminder, exact finite difference schemes can be characterized as follows [8]: Let $u(x, y, t)$ be the solution to a two space dimension PDE; let $u^k_{m,n}$ be the solution to a finite difference model of the PDE such that

$$t_k = (\Delta t)k, \qquad x_m = (\Delta x)m, \qquad y_n = (\Delta y)n. \tag{4}$$

Then the finite difference scheme is exact if

$$u(x_m, y_n, t_k) = u^k_{m,n} \tag{5}$$

for all (positive) values of the step sizes $(\Delta t, \Delta y, \Delta t)$. In general, functional relations will exist among the three step-sizes; for example

$$\Delta x = g_1(\Delta t), \qquad \Delta y = g_2(\Delta t), \tag{6}$$

where g_1 and g_2 are known functions. Thus, subject to the constraints of equations (6), such schemes give the exact value of $u(x, y, t)$ at each point on the computational grid [8]; the truncation error is zero!

To proceed, the exact solutions of equations (1) and (2) must be determined. A direct calculation shows that equation (1) has the solution

$$u(x, y, t) = f(x - at, y - bt), \tag{7}$$

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if $u(x, y, 0) = f(x, y)$ is given and f has a first derivative [9]. Likewise, for equation (2), if $u(x, y, 0) = g(x, y)$ is given, where for physical reasons $0 \leqslant f(x, y) \leqslant 1$, then the general solution is

$$u(x, y, t) = g(x - at, y - bt)/[(1 - e^{-t})g(x - at, y - bt) + e^{-t}]. \tag{8}$$

To derive this result, equation (2) is linearized by means of the transformation

$$u(x, y, t) = 1/w(x, y, t) \tag{9}$$

to give

$$w_t + aw_x + bw_y + w = 1. \tag{10}$$

This last equation can be solved exactly [9].

Now observe that if Δx and Δy are selected so that

$$\Delta x = a\Delta t, \qquad \Delta y = b\Delta t, \tag{11}$$

and if $u^k_{m,n}$ is taken to be

$$u^k_{m,n} \equiv f(x_m - at_k, y_n - bt_k) = F(m - k, n - k), \tag{12}$$

then $u^k_{m,n}$ satisfies the partial difference equation

$$u^{k+1}_{m,n} = u^k_{m-1,m-1}. \tag{13}$$

This is the exact finite difference scheme for equation (1). A little algebraic manipulation will put equation (13) into the form

$$\frac{u^{k+1}_{m,n} - u^k_{m,n}}{\Delta t} + \left(\frac{a}{\Delta x}\right)\left[\left(\frac{u^k_{m,n} + u^k_{m,n-1}}{2}\right) - \left(\frac{u^k_{m-1,n} + u^k_{m-1,n-1}}{2}\right)\right]$$
$$+\left(\frac{b}{\Delta y}\right)\left[\left(\frac{u^k_{m-1,n} + u^k_{m,n}}{2}\right) - \left(\frac{u^k_{m-1,n-1} + u^k_{m,n-1}}{2}\right)\right] = 0. \tag{14}$$

Examination of this expression shows that the discrete time derivative is the usual forward Euler method. However, the discrete space derivatives have a more complex form. For example, the discrete x derivative is the backward Euler method applied to the average of $u^k_{m,n}$ over the n variable using adjacent lattice points in the m variable. A similar situation holds for the discrete y derivative.

The same analysis can be applied to equation (2) and its solution given by equation (8). The way to proceed is to solve equation (8) for $g(x - at, y - bt)$ and use equation (13). The corresponding exact finite difference scheme for equation (2) is

$$u^{k+1}_{m,n} = u^k_{m-1,n-1}/[e^{-\Delta t} + (1 - e^{-\Delta t})u^k_{m-1,n-1}], \tag{15}$$

which can be rewritten in the form

$$(u^{k+1}_{m,n} - u^k_{m,n})/\phi_1(\Delta t) + a[(u^k_{m,n} - u^k_{m-1,n})/\phi_2(\Delta x)] + b[(u^k_{m-1,n} - u^k_{m-1,n-1})/\phi_3(\Delta y)]$$
$$= u^k_{m-1,n-1}(1 - u^{k+1}_{m,n}), \tag{16}$$

where

$$\phi_1(\Delta t) = e^{\Delta t} - 1, \qquad \phi_2(\Delta x) = a(e^{\Delta x/a} - 1), \qquad \phi_3(\Delta y) = b(e^{\Delta y/b} - 1), \quad \text{(17a, b)}$$

and the functional relations given by equation (11) hold.

It is clear that the exact schemes for equations (1) and (2) given, respectively, by equations (14) and (16) are not what is obtained using the standard finite difference methods [2, 4, 5]. For example, a standard scheme might give the following discrete model for equation (2)

$$(u_{m,n}^{k+1} - u_{m,n}^{k})/\Delta t + a([u_{m+1,n}^{k} - u_{m-1,n}^{k}]/\Delta x) + b([u_{m,n+1}^{k} - u_{m,n-1}^{k}]/\Delta y) = u_{m,n}^{k}\,(1 - u_{m,n}^{k}). \qquad (18)$$

with no *a priori* functional relations holding between the step-sizes. This scheme will, in general, have "numerically choatic" solutions [8]. Other standard schemes for both equations (1) and (2) will lead to similar results. However, the schemes of this letter are exact and give the restrictions that must hold among the step-sizes. Consequently, these schemes can be used as bench marks to test finite difference models for more complex PDE's [10].

The final observation is that the exact finite difference scheme for equations (1) and (2), given, respectively, by equations (13) and (15), are quite simple in their structures. They both are explicit schemes and easy to use for the calculation of numerical solutions. One begins by selecting a value for Δt; the values for Δx and Δy are then given by equation (11). For a particular initial function, $u_{n,m}^{0}$, the numerical solution at any discrete time t_k can then be determined in a straight forward manner. The sole reason for rewriting these simple expressions to the complex forms of equations (14) and (16) is to have relations that can be directly compared to standard finite difference schemes such as equation (18).

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